# Energy-momentum tensor of bouncing gravitons

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**Abstract**

In models of the Universe with extra dimensions gravity propagates in the whole space-time. Graviton production by matter on the brane is significant in the early hot Universe. In a model of 3-brane with matter embedded in 5D space-time conditions for gravitons emitted from the brane to the bulk to return back to the brane are found. For a given 5-momentum of graviton falling back to the brane the interval between the times of emission and return to the brane is calculated. A method to calculate contribution to the energy-momentum tensor from multiple graviton bouncings is developed. Explicit expressions for contributions to the energy-momentum tensor of gravitons which have made one, two and three bounces are obtained and their magnitudes are numerically calculated. These expressions are used to solve the evolution equation for dark radiation. A relation connecting reheating temperature and the scale of extra dimension is obtained. For the reheating temperature $T_R \sim 10^6 GeV$ we estimate the scale of extra dimension $\mu$ to be of order $10^{-9} GeV$ $(\mu^{-1} \sim 10^{-5} cm)$.

## 1 Introduction

Brane-world scenarios with the observable Universe located on a 3-brane embedded in a higher-dimensional space-time have attracted considerable interest recently. Such models with matter on the brane can reproduce the main cosmological data [1, 2, 3, 4].

A general property of extra-dimensional models is that although ordinary matter is supposed to be confined to a brane, gravity propagates in the whole space-time. This entails the effect that gravitons produced in reactions of particles on the brane can escape to the bulk. Graviton production is strong in the early hot Universe, and can alter the time evolution of matter on the brane and, in particular, the primordial nucleosynthesis.

In this paper we calculate graviton production in a model of five-dimensional Universe with one large extra dimension. Matter is supposed to be confined to the 3D brane. Time evolution of matter in this model is described by the generalized Friedmann equation $H^2 = \rho^2 + 2\mu\rho + \cdots$ [5, 6, 7]. We consider the period of early cosmology, in which the term quadratic in energy density is dominant, $\mu/\rho \ll 1$ ($\rho$ is the normalized energy density on the brane defined in (4), $\mu = (-\Lambda/6)^{1/2}$, and $\Lambda$ is 5D cosmological constant).

Because the space-time is curved, a part of gravitons emitted in the bulk can return back to the brane [8, 10, 11] and bounce again to the bulk. In paper [12] an analytical method to show that a bounce is possible was developed. In the present paper we investigate further conditions of a bounce. Solving the combined system of equations of trajectories of the brane and of emitted graviton, we find conditions at which graviton can fall back to the brane. We derive an equation for the interval

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of time between graviton emission and its return to the brane and develop a scheme to calculate times of returns of graviton to the brane for multiple bounces. We show that in the period of early cosmology the ratio of times $t_0/t_1$ of graviton emission $t_0$ and its return to the brane $t_1$ to a good approximation can be expressed as a function of $x = m(t_1)/E(t_1)$, where $(E(t_1), m(t_1), \mathbf{p})$ are the components of graviton 5-momentum at the time $t_1$ when graviton returns to the brane.

As an application of the above results, using the distribution function of emitted gravitons of paper [11], we calculate the the components of the energy-momentum tensor of bouncing gravitons $T_{nn}^{in,(k)}$ ($n^A$ is transverse to the brane, k is a number of a bounce). For the first three bounces we obtain explicit expressions for $T_{nn}^{in,(k)}$ and estimate their numerical magnitudes. The expressions for the energy-momentum tensor of bouncing gravitons are used to solve the evolution equation of dark radiation [11, 13]. Solving this equation, we find a relation connecting the reheating temperature of the Universe $T_R$ and the scale of the extra dimension $\mu$. Qualitative constraints on $T_R$ and $\mu$ are discussed.

In Sect.2 we review two approaches to the 5D model.

In Sect.3 we solve geodesic equations for gravitons propagating in the bulk.

In Sect.4 we solve the combined system of equations for graviton and brane trajectories and find conditions for return of graviton to the brane. We calculate the interval of times between graviton emission and detection as a function of graviton momentum at the time of detection.

In Sect. 5 consider multiple graviton bounces. We calculate the $(nn)$ components of the energy-momentum tensor of gravitons falling to the brane.

In Sect. 6 we present qualitative numerical analysis of the energy-momentum tensor and discuss solution of evolution equation for dark radiation.

## 2 3-brane in 5D bulk

We consider the 5D model with one 3D brane embedded in the bulk. Matter is confined to the brane, gravity extends to the bulk. In the leading approximation we neglect graviton emission from the brane to the bulk. The action is taken in the form

$$S_5 = \frac{1}{2\kappa^2}\left[\int_{\Sigma} d^5x\sqrt{-g^{(5)}}(R^{(5)} - 2\Lambda) + 2\int_{\partial\Sigma} K\right] - \int_{\partial\Sigma} d^4x\sqrt{-g^{(4)}}\,\hat{\sigma} - \int_{\partial\Sigma} d^4x\sqrt{-g^{(4)}}L_m, \quad (1)$$

where $x_4 \equiv y$ is coordinate of the infinite extra dimension, $\kappa^2 = 8\pi/M^3$ .

The 5D model can be treated in two alternative approaches. In the first approach metric is non-static, and the brane is located at a fixed position in the extra dimension [5, 6]. We consider the class of metrics of the form

$$ds_5^2 = g_{AB}^{(5)}dx^Adx^B = -n^2(y,t)dt^2 + a^2(y,t)\eta_{ab}dx^adx^b + dy^2 \quad (2)$$

The brane is spatially flat and located at $y = 0$. The freedom of parametrization of $t$, allows to set $n(0,t) = 0$. The energy-momentum tensor of matter on the brane is taken in the form

$$\hat{t}_\mu^\nu = diag\,\delta(y)\{-\hat{\rho}, \hat{p}, \hat{p}, \hat{p}\}. \quad (3)$$

For the following it is convenient to introduce the normalized expressions for energy density, pressure and cosmological constant on the brane which all have the same dimensionality $[GeV]$

$$\mu = \sqrt{-\frac{\Lambda}{6}}, \quad \sigma = \frac{\kappa^2\hat{\sigma}}{6}, \quad \rho = \frac{\kappa^2\hat{\rho}}{6}, \quad p = \frac{\kappa^2\hat{p}}{6}. \quad (4)$$

Reduction of the metric (2) to the brane is

$$ds^2 = dt^2 + a^2(0,t)\eta_{ab}dx^a dx^b. \tag{5}$$

The function $a(t) = a(0,t)$ satisfies the generalized Friedmann equation [6]

$$H^2(t) = -\mu^2 + (\rho+\sigma)^2 + \mu\rho_w(t), \tag{6}$$

where $H(t) = \dot{a}(t)/a(t)$ and $\rho_w(t)$ is the Weyl radiation term [1], which below is set to zero.

In the second approach the brane separates two static 5D AdS spaces attached to both sides of the brane. The metrics of the AdS spaces are solutions of the Einstein equations of the form

$$ds^2 = -f_i(R)d\tilde{T}^2 + \frac{dR^2}{f_i(R)} + \mu_i^2 R^2 dx^a dx_a, \tag{7}$$

where

$$f_i(R) = \mu_i^2 R^2 - \frac{P_i}{R^2}.$$

Below we consider the case $\mu_1 = \mu_2$ and $P_i = 0$. Trajectory of the moving brane in the $R, T$ plane is given by parametric equations $R = r_b(t)$, $\tilde{T} = \tau_b(t)$, where $t$ is the proper time on the brane

$$-f(r_b)\dot{\tau}_b^2 + f^{-1}(r_b)\dot{r}_b^2 = -1, \tag{8}$$

dot is derivative over $t$. Reduction of the 5D metric to the brane is

$$ds^2 = -dt^2 + r_b^2(t)dx^a dx_a. \tag{9}$$

The function $r_b(t)$ satisfies the generalized Friedmann equation [14, 15, 16]

$$\left(\frac{\dot{r}_b}{r_b}\right)^2 = -\mu^2 + (\rho+\sigma)^2. \tag{10}$$

Equations (10) and (6) with $\rho_w(t) = 0$ are of the same form, and $a^2(0,t)$ can be identified with $r_b^2(t)$. Below we consider the case $\sigma = \mu$ [4], so that (10) takes a form

$$\left(\frac{\dot{r}_b}{r_b}\right)^2 = \rho^2 + 2\mu\rho. \tag{11}$$

The normalized velocity vector of the brane and the normal vector to the brane are

$$v^A = (v^T,\, v^R) = (\dot{\tau}_b,\, \dot{r}_b), \qquad n^A = (n^T,\, n^R) = \pm\left(\frac{\dot{r}_b}{f(r_b)},\, f(r_b)\dot{\tau}_b\right) \tag{12}$$

Here

$$\dot{\tau}_b = \epsilon\frac{\sqrt{f+\dot{r}_b^2}}{f(r_b)}, \tag{13}$$

where $\epsilon = \pm$.

In the following we choose $n^A$ with the sign (-)

$$n^A = -\left(\frac{\dot{r}_b}{f(a)},\, f(a)\dot{\tau}_b\right).$$

# 3 Geodesic equations in the picture with static metric

Let $\lambda$ be parameter along a geodesic. Geodesic equations in the metric (7) are

$$\frac{d^2\tilde{T}}{d\lambda^2} + 2\Gamma^T_{TR}\frac{d\tilde{T}}{d\lambda}\frac{dR}{d\lambda} = 0 \quad (14)$$

$$\frac{d^2x^a}{d\lambda^2} + 2\Gamma^a_{bR}\frac{dx^b}{d\lambda}\frac{dR}{d\lambda} = 0 \quad (15)$$

$$\frac{d^2R}{d\lambda^2} + \Gamma^R_{RR}\left(\frac{dR}{d\lambda}\right)^2 + \Gamma^R_{TT}\left(\frac{d\tilde{T}}{d\lambda}\right)^2 + \Gamma^R_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = 0, \quad (16)$$

We consider solutions of the geodesic equations even in $\lambda$. Integrating the geodesic equations, one obtains [13]

$$\frac{d\tilde{T}}{d\lambda} = \frac{C^T}{\mu^2R^2}, \qquad \frac{dx^a}{d\lambda} = \frac{C^a}{\mu^2R^2}, \qquad \left(\frac{dR}{d\lambda}\right)^2 = \mu^2R^2\,(C^R)^2 + C^{T2} - C^{a2}, \quad (17)$$

where $(C^T,\, C^a,\, C^R)$ are integration parameters.

Tangent vectors to a null geodesic satisfy the relation

$$g_{AB}\frac{dx^A}{d\lambda}\frac{dx^B}{d\lambda} = 0, \quad (18)$$

from which it follows that $C^R = 0$. Eqs. (14)-(17) were solved with the initial condition that $\tilde{T}(0)$ and $R(0)$ are located on the brane world sheet: $\tilde{T}(0) = \tau_b(t_0)$, $R(0) = r_b(t_0)$. Here $t_0$ is the proper time of the point on the brane world sheet $\tau_b(t_0)$, $r_b(t_0)$ at which the geodesic begins, i.e. the time of the graviton emission.

The components of momentum of a graviton propagating along a null geodesic are proportional to the tangent vector to a null geodesic

$$(p^T,\, p^R,\, p^a) \sim \left(\frac{C^T}{\mu^2R^2},\, \epsilon_R(C^{T2} - C^{a2})^{1/2},\, \frac{C^a}{\mu^2R^2}\right), \quad (19)$$

where $\epsilon_R = \pm$. Also we define $\epsilon_T$ as $C^T = \epsilon_T|C^T|$. Expanding the graviton momentum $p^A$ in the basis $(v^A(t), n^A(t), e^A_{\bar{a}}(t))$, where $e^A_{\bar{a}} = \delta^A_{\bar{a}}/\mu a$, we have

$$p^A = Ev^A + mn^A + p^{\bar{a}}e^A_{\bar{a}}, \quad (20)$$

where

$$p^Ap_A = -E^2 + m^2 + p^{\bar{a}2} = 0. \quad (21)$$

The components of $p^A$ in the two bases are connected as

$$p^T = \frac{E\epsilon\sqrt{\mu^2+H^2} - m\,H}{\mu^2r_b}, \quad p^R = r_b(EH - m\epsilon\sqrt{\mu^2+H^2}), \quad p^a = \frac{p^{\bar{a}}}{\mu r_b} \quad (22)$$

$$E = p^T\epsilon\sqrt{\mu^2+H^2}r_b - \frac{p^RH}{\mu^2r_b}, \quad m = p^THr_b - p^R\frac{\epsilon\sqrt{\mu^2+H^2}}{\mu^2r_b}. \quad (23)$$

The components $E$ and $m$ depend on $t$ through $r_b(t)$.

Introducing

$$\gamma = \frac{C^{a2}}{C^{T2}} = 1 - \frac{C^{R2}}{C^{T2}}, \tag{24}$$

and expressing $\gamma$ through $E$ and $m$, we have

$$\gamma = 1 - \frac{1}{(\mu^2 R^2)^2}\frac{p^{R2}}{p^{T2}} = 1 - \left(\frac{EH - m\epsilon\sqrt{\mu^2+H^2}}{E\epsilon\sqrt{\mu^2+H^2} - mH}\right)^2 \frac{r_b^4(t)}{R^4}. \tag{25}$$

If at a time $t$ graviton is on the brane world sheet $R = r_b(t)$ ($t$ is a time of emission, $t_0$, or a time of return of the graviton to the brane, $t_1$, we take the basis $(v^A, n^A, e^A)(t)$ at the time $t$ and obtain $\gamma$ in a form

$$\gamma = \frac{\mu^2(E^2 - m^2)}{(E\epsilon\sqrt{\mu^2+H^2} - mH)^2}. \tag{26}$$

# 4 Bounce of massless particles in the period of early cosmology

We consider the radiation-dominated period of the early cosmology, when $\rho/\mu \gg 1$ or,equivalently, $\mu t \ll 1$. Supposing that the energy loss from the brane to the bulk is sufficiently small to comply with the observational data, we neglect in the conservation equation for the energy-momentum tensor the energy flow in the bulk. In the period of early cosmology, in the model with extra dimension, from the expression for energy density of relativistic degrees of freedom

$$\rho(T) = \frac{\kappa^2\pi^2 g_*(T)\,T^4}{180}, \tag{27}$$

($g_*(T)$ is a total number of relativistic degrees of freedom [18, 19] ) it follows that

$$\frac{\rho(t)}{\rho(t_1)} = \frac{g_*(T)T^4}{g_*(T_1)T_1^4} \simeq \frac{T^4}{T_1^4} \tag{28}$$

For times $t_1$ and $t$ in the region of early cosmology, from the Friedman equation one obtains

$$\frac{\rho(t)}{\rho(t_1)} = \left(\frac{r_b(t_1)}{r_b(t)}\right)^4 \simeq \frac{t_1}{t}. \tag{29}$$

Following [11], with the use of (8), the equation for the brane trajectory can be written as

$$\frac{dr_b}{d\tau_b} = \epsilon\frac{\mu^2 r_b^2 \dot{r}_b}{\sqrt{\mu^2 r_b^2 + \dot{r}_b^2}} = \epsilon\mu^2 r_b^2 \frac{H}{\sqrt{\mu^2+H^2}}. \tag{30}$$

Integrating Eq. (30) with the boundary conditions $r_b = r_b(t_0)$, $\tau_b = \tau_b(t_0)$, we obtain the equation for trajectory of the brane

$$\mu^2(\tau_b(t) - \tau_b(t_0)) = \epsilon\int_{r_b(t_0)}^{r_b(t)} \frac{dr}{r^2}\frac{\sqrt{\mu^2+H^2}}{H} = \epsilon\left(\frac{1}{r_b(t_0)} - \frac{1}{r_b(t)}\right) + \epsilon\int_{r_b(t_0)}^{r_b(t)} \frac{dr}{r^2}\left[\frac{\sqrt{\mu^2+H^2}}{H} - 1\right]. \tag{31}$$

From the first integrals of the null geodesic equations (17) we obtain

$$\frac{dR}{d\tilde{T}} = \epsilon_T \epsilon_R (1-\gamma)^{1/2} \mu^2 R^2. \tag{32}$$

Integrating Eq. (32) with the initial conditions $R = r_b(t_0)$, $\tilde{T} = \tau_b(t_0)$, we obtain the equation for for a null geodesic (graviton trajectory)

$$\frac{1}{r_b(t_0)} - \frac{1}{R} = \epsilon_T \epsilon_R (1-\gamma)^{1/2} \mu^2 (\tilde{T} - \tau_b(t_0)). \tag{33}$$

If graviton returns to the brane at time $t_1$, we have $R = r_b(t_1)$. Combining Eqs. (31) and (33) and using Friedmann equation, $H^2 = \rho^2 + 2\mu\rho$, we obtain an equation for $r_b(t_1)$

$$[\epsilon \epsilon_T \epsilon_R (1-\gamma)^{-1/2} - 1] \left( \frac{1}{r_b(t_0)} - \frac{1}{r_b(t_1)} \right) = \int_{r_b(t_0)}^{r_b(t_1)} \frac{dr}{r^2} \left[ \frac{\rho + \mu}{\sqrt{\rho^2 + 2\mu\rho}} - 1 \right]. \tag{34}$$

Eq. (34) can be interpreted as an equation which determins the time of return of graviton to the brane $t_1$ for a given time of emission $t_0$. It is seen that Eq.(34) admits solution only if

$$\epsilon \epsilon_T \epsilon_R = (+) \tag{35}$$

Expanding the integrand of (34) in powers of $\mu/\rho$, we have

$$\frac{\rho + \mu}{\sqrt{\rho^2 + 2\rho\mu}} = \left( 1 + \frac{1}{2} \left( \frac{\mu}{\rho} \right)^2 - \left( \frac{\mu}{\rho} \right)^3 + \frac{15}{8} \left( \frac{\mu}{\rho} \right)^4 + ... \right),$$

where $\rho(t) = \rho(t_0)(r_b(t_0)/r_b(t))^4$. Integrating Eq. (34), we obtain

$$\begin{aligned} &((1-\gamma)^{-1/2} - 1) \left( \frac{1}{r_b(t_0)} - \frac{1}{r_b(t_1)} \right) \\ &= \frac{1}{r_b(t_0)} \left[ \frac{1}{14} \left( \frac{\mu}{\rho_0} \right)^2 \left( \left( \frac{r_b(t_1)}{r_b(t_0)} \right)^7 - 1 \right) - \frac{1}{11} \left( \frac{\mu}{\rho_0} \right)^3 \left( \left( \frac{r_b(t_1)}{r_b(t_0)} \right)^{11} - 1 \right) + ... \right]. \end{aligned} \tag{36}$$

The series in (36) is convergent. Introducing

$$z = \frac{r_b(t_0)}{r_b(t_1)} \simeq \left( \frac{t_0}{t_1} \right)^{1/4} \tag{37}$$

and substituting $\rho_0 = \rho_1 z^{-4}$, where $\rho_0 = \rho(t_0)$ and $\rho_1 = \rho(t_1)$, we transform (36) to a form

$$[(1-\gamma)^{-1/2} - 1] = \frac{1}{2} \left( \frac{\mu}{\rho_1} \right)^2 \frac{z(1-z^7)}{7(1-z)} - \left( \frac{\mu}{\rho_1} \right)^3 \frac{z(1-z^{11})}{11(1-z)} + .... \tag{38}$$

The function

$$f_k(z) = \frac{z(1-z^k)}{k(1-z)} \tag{39}$$

is monotone increasing with the maximum at the point $z = 1$ equal to 1.

### 4.1 Conditions of the fall of graviton on the brane

The sign of the graviton momentum component $m(t)$ at the time $t_1$ at which graviton returns to the brane is opposite to that at the time of emission $t_0$. From (23) we have

$$\begin{aligned} E &\sim \epsilon_T\sqrt{H^2+\mu^2} - \epsilon_R H(1-\gamma)^{1/2} \\ m &\sim \epsilon_T H - \epsilon_R\sqrt{H^2+\mu^2}(1-\gamma)^{1/2}. \end{aligned} \tag{40}$$

Condition (35) is satisfied in the following cases:

$$\begin{array}{ll} \text{(i) } \epsilon_T=+,\ \ \epsilon_R=+;\ \ \epsilon=+; & \text{(ii)}\epsilon_T=-,\ \ \epsilon_R=-;\ \ \epsilon=+; \\ \text{(iii)}\epsilon_T=-,\ \ \epsilon_R=+;\ \ \epsilon=-; & \text{(iv)}\epsilon_T=+,\ \ \epsilon_R=-;\ \epsilon=- \end{array}$$

Let us find in which case is realized one of the possibilities: either $(a)$ $m_0 = m(\tau_0) > 0,\ \ m_1 = m(t_1) < 0$, or $(b)$ $m_0 < 0,\ \ m_1 > 0$.

- The case (i)(a). $\epsilon_T = \epsilon_R = \epsilon = +$ .

  $E$ and $m$ are

  $$E(t) \sim \sqrt{H^2+\mu^2} - H(1-\gamma)^{1/2} > 0, \qquad m(t) \sim H - \sqrt{H^2+\mu^2}(1-\gamma)^{1/2}.$$

  Conditions

  $$m(\tau_0) \sim H_0 - \sqrt{H_0^2+\mu^2}(1-\gamma)^{1/2} > 0, \qquad m(t_1) \sim H_1 - \sqrt{H_1^2+\mu^2}(1-\gamma)^{1/2} < 0.$$

  are satisfied, if

  $$\frac{H_0}{\sqrt{\mu^2+H_0^2}} > (1-\gamma)^{1/2} > \frac{H_1}{\sqrt{\mu^2+H_1^2}}. \tag{41}$$

  Here $H_0 = H(t_0),\ H_1 = H(t_1)$.

- The case (i)(b). $\epsilon_T = \epsilon_R = \epsilon = +$.

  $E$ and $m$ are

  $$E(t) \sim \sqrt{H^2+\mu^2} - H(1-\gamma)^{1/2} > 0 \qquad m(t) \sim H - \sqrt{H^2+\mu^2}(1-\gamma)^{1/2}.$$

  From conditions $m_0 < 0,\ \ m_1 > 0$ it follows that

  $$\frac{H_0}{\sqrt{\mu^2+H_0^2}} < (1-\gamma)^{1/2} < \frac{H_1}{\sqrt{\mu^2+H_1^2}}$$

  or $H_0 < H_1$, which is impossible, because $t_0 < t_1$.

- The case (ii)(a). $\epsilon_T = \epsilon_R = -,\ \ \epsilon = +$ .

  Analogously to the case (i)(b) in the case (ii)(a) there are no solutions.

- The case (ii)(b). $\epsilon_T = \epsilon_R = -,\ \ \epsilon = +$ .

  $E$ and $m$ are

  $$E(t) \sim -\sqrt{H^2+\mu^2} + H(1-\gamma)^{1/2} < 0 \qquad m(t) \sim -H + \sqrt{H^2+\mu^2}(1-\gamma)^{1/2}.$$

  Solution $m_0 < 0,\ m_1 > 0$ exists, provided (41) is valid.

- The case (iii)(a). $\epsilon_T = -, \ \ \epsilon_R = +$.

  $E$ and $m$ are

  $$E(t) \sim -\sqrt{H^2+\mu^2} - H(1-\gamma)^{1/2} \qquad m(t) \sim -H - \sqrt{H^2+\mu^2}(1-\gamma)^{1/2} < 0.$$

  Because $m$ is negative and does not change its sign, there are no solutions. Analogously in the case (iv) $m$ is always positive, and no solution exists.

To conclude, we are left with the solutions of the types (i)(a) and (ii)(b), which are physically equivalent, because Eqs. (17) with $\epsilon_T = \epsilon_R = +$ transform to equations with $\epsilon_T = \epsilon_R = -$ under the change $\lambda \to -\lambda$. In the following we consider the case (i)(a).

### 4.2 Relation between emission and detection times

To solve Eq.(38) we need to transform the expression $(1-\gamma)^{-1/2}-1$ to a convenient form. Introducing $x = E/m$ and using Friedman equation, $H^2 = \rho^2 + 2\mu\rho$, and (26), we have

$$(1-\gamma)^{-1/2} - 1 = \left| \frac{\sqrt{H^2+\mu^2} - xH}{H - x\sqrt{H^2+\mu^2}} \right| - 1 = \left| \frac{1+\mu/\rho - x\sqrt{1+2\mu/\rho}}{\sqrt{1+2\mu/\rho} - x(1+\mu/\rho)} \right| - 1 \tag{42}$$

Relation (42) is valid at the endpoints of graviton trajectory, at emission point and at points where graviton hits the brane.

Because, as discussed in preceding subsection, $x_1 < 0$, the relation (42) written at time $t_1$ takes the form

$$(1-\gamma)^{-1/2} - 1 = (1-|x_1|)\frac{1+\mu/\rho_1 - \sqrt{1+2\mu/\rho_1}}{|x_1|(1+\mu/\rho_1) + \sqrt{1+2\mu/\rho_1}}. \tag{43}$$

In the period of early cosmology expression (43) approximately is

$$(1-\gamma)^{-1/2} - 1 \simeq (1-|x_1|)\frac{\mu^2/2\rho_1^2}{\sqrt{1+2\mu/\rho_1} + |x_1|(1+\mu/\rho_1)}. \tag{44}$$

The advantage of this form of $\gamma$ is that we have extracted the factor $(\mu/\rho_1)^2$. Now the Eq. (38) can be written as

$$\left(\frac{\mu}{\rho_1}\right)^2 \frac{1-|x_1|}{(1+|x_1|)(1+\mu/\rho_1)} = \left(\frac{\mu}{\rho_1}\right)^2 \left[ f_7(z_{01}) - \left(\frac{\mu}{\rho_1}\right) 2f_{11}(z_{01}) + ... \right], \tag{45}$$

or

$$\frac{1-|x_1|}{(1+|x_1|)} = \left[ f_7(z_{01}) - \left(\frac{\mu}{\rho_1}\right) 2f_{11}(z_{01}) + ... \right](1+\mu/\rho_1), \tag{46}$$

Eqs. (38) define $z_{01} \simeq (t_0/t_1)^{1/4}$ through the ratio $x_1 = m_1/E_1$ and $\rho_1 = \rho(t_1)$, or, equivalently, the emission time $t_0$ through the "mass" and "energy" of the graviton at the time of return of the graviton to the brane $t_1$. In the following, for practical calculations, in the region $\mu/\rho_1 < 1$ we use a simplified equation

$$\frac{1-|x_1|}{1+|x_1|} \simeq f_7(z_{01}). \tag{47}$$

Domain of applicability and corrections to this equation are discussed in Sect.6.

## 5 Multiple bounces

To consider multiple reflections from the brane of bouncing gravitons we use matrix notations. Introducing

$$K = K^{-1} = \left| \begin{array}{cc} \sqrt{H^2/\mu^2+1} & -H/\mu \\ H/\mu & -\sqrt{H^2/\mu^2+1} \end{array} \right|, \tag{48}$$

and

$$\tilde{p}^T = p^T \mu R, \qquad \tilde{p}^R = \frac{p^R}{\mu R}, \quad p^{\bar{a}} = \mu R p^a, \tag{49}$$

we have

$$\begin{pmatrix} \tilde{p}^T \\ \tilde{p}^R \end{pmatrix} = K \begin{pmatrix} E \\ m \end{pmatrix}, \qquad \begin{pmatrix} E \\ m \end{pmatrix} = K \begin{pmatrix} \tilde{p}^T \\ \tilde{p}^R \end{pmatrix}. \tag{50}$$

The case with multiple bouncings is illustrated by the scheme

$$\begin{pmatrix} p_0 \\ E_0 \\ m_0 \end{pmatrix}^{out} (t_0) \overset{z_{01}}{\to} \begin{pmatrix} p_1^{in} = p_1^{out} \\ E_1^{in} = E_1^{out} \\ m_1^{in} = -m_1^{out} \end{pmatrix} (t_1) \overset{z_{12}}{\to} \begin{pmatrix} p_2^{in} = p_2^{out} \\ E_2^{in} = E_2^{out} \\ m_2^{in} = -m_2^{out} \end{pmatrix} (t_2) \overset{z_{23}}{\to} \cdots \tag{51}$$

The left column corresponds to the emission time $t_0$. In the next brackets, in the left columns are momenta of ingoing particle, in the right ones are the outgoing. Under reflection from the brane momentum $p^a$ parallel to the brane and energy $E$ are conserved, transverse momentum to the brane $m$ changes its sign:

$$E \to E, \qquad p^a \to p^a, \qquad m \to -m.$$

Momenta $\tilde{p}^{T,R} \sim C^{T,R}/r_b(t)$ are are rescaled when moving from one bracket to the next

$$\frac{\tilde{p}_n^{T,R}}{\tilde{p}_{n-1}^{T,R}} = \frac{r_b(t_{n-1})}{r_b(t_n)} \simeq \left(\frac{t_{n-1}}{t_n}\right)^{1/4} = z_{n-1,n}. \tag{52}$$

Transformation from "in" to "out" components within a bracket is given by

$$M^{out} \equiv \begin{pmatrix} E^{out} \\ m^{out} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} E^{in} \\ m^{in} \end{pmatrix} \equiv L M^{in}. \tag{53}$$

Let us consider the first two brackets in (51). Using (50) we express $E_0$ and $m_0$ through $E_1$ and $m_1$ and obtain

$$\begin{pmatrix} E_0 \\ m_0 \end{pmatrix}^{out} = z_{01}^{-1} K_0 K_1 \begin{pmatrix} E_1 \\ m_1 \end{pmatrix}^{in} \tag{54}$$

$$= z_{01}^{-1} \begin{pmatrix} \sqrt{\frac{H_0^2}{\mu^2}+1}\sqrt{\frac{H_1^2}{\mu^2}+1} - \frac{H_0 H_1}{\mu^2} & \frac{H_0}{\mu}\sqrt{\frac{H_1^2}{\mu^2}+1} - \frac{H_1}{\mu}\sqrt{\frac{H_0^2}{m^2}+1} \\ \frac{H_0}{\mu}\sqrt{\frac{H_1^2}{\mu^2}+1} - \frac{H_1}{\mu}\sqrt{\frac{H_0^2}{\mu^2}+1} & \sqrt{\frac{H_0^2}{\mu^2}+1}\sqrt{\frac{H_1^2}{\mu^2}+1} - \frac{H_0 H_1}{\mu^2} \end{pmatrix} \begin{pmatrix} E_1 \\ m_1 \end{pmatrix}.$$

In the period of early cosmology, from the Friedman equation it follows that $H/\mu \simeq 1/(4\mu t)$. For small $\mu t$ we can simplify the expressions in (54) as

$$\sqrt{\frac{H_0^2}{\mu^2}+1}\sqrt{\frac{H_1^2}{\mu^2}+1}-\frac{H_0H_1}{\mu^2}\simeq\frac{1}{2}\left(\frac{H_0}{H_1}+\frac{H_1}{H_0}\right)\simeq\frac{1}{2}\left(\frac{t_1}{t_0}+\frac{t_0}{t_1}\right)=\frac{z_{01}^{-4}+z_{01}^4}{2} \tag{55}$$
$$\frac{H_0}{\mu}\sqrt{\frac{H_1^2}{\mu^2}+1}-\frac{H_1}{\mu}\sqrt{\frac{H_0^2}{\mu^2}+1}\simeq\frac{1}{2}\left(\frac{H_0}{H_1}-\frac{H_1}{H_0}\right)\simeq\frac{1}{2}\left(\frac{t_1}{t_0}-\frac{t_0}{t_1}\right)=\frac{z_{01}^{-4}-z_{01}^4}{2}.$$

Introducing

$$\psi^{\pm}(z)=\frac{z^{-4}\pm z^4}{2},$$

we obtain

$$\begin{pmatrix} E_0 \\ m_0 \end{pmatrix}^{out} = z_{01}^{-1}\begin{pmatrix} \psi_{01}^+ & \psi_{01}^- \\ \psi_{01}^- & \psi_{01}^+ \end{pmatrix}\begin{pmatrix} E_1 \\ m_1 \end{pmatrix}^{in}, \tag{56}$$

where

$$\psi_{01}^{\pm}=\psi^{\pm}(z_{01})$$

For multiple bouncings we have

$$\begin{aligned} M_0^{out} &= z_{01}^{-1}K_0K_1M_1^{in}, \\ M_0^{out} &= z_{02}^{-1}K_0\,K_1LK_1\,K_2M_2^{in}, \\ \cdots \\ M_0^{out} &= Z_{0,n}^{-1}K_0\,K_1LK_1\,...K_{n-1}K_nM_n^{in}, \end{aligned} \tag{57}$$

where

$$z_{0n}=z_{01}\,z_{12}\,z_{23}\cdots z_{n-1,n}, \tag{58}$$

$$K_0\,K_1LK_1\,...K_{n-1}K_n=\begin{pmatrix} \psi_{0n}^+ & \psi_{0n}^- \\ (-)^{n+1}\psi_{0n}^- & (-)^{n+1}\psi_{0n}^+ \end{pmatrix}. \tag{59}$$

Here

$$\psi_{0n}^{\pm}=\psi^{\pm}(u_{0n}), \tag{60}$$

$$u_{0n}=u_{0,n-1}^{-1}\,z_{n-1,n}, \tag{61}$$

or explicitly

$$u_{02}=z_{01}^{-1}\,z_{12},\quad u_{03}=z_{01}\,z_{12}^{-1}\,z_{23},\quad u_{04}=z_{01}^{-1}\,z_{12}\,z_{23}^{-1}z_{34},\,\cdots$$

From (60) and (61) it follows that $u_{0,2k+1}<1$ and $u_{0,2k}>1$ . In the latter case

$$\psi_{0,2k}^-(u_{0,2k})=-|\psi_{0,2k}^-(u_{0,2k})|=-\psi_{0,2k}^-(u_{0,2k}^{-1}), \tag{62}$$

and

$$E_0=z_{0,2k}(\psi_{0,2k}^+E_{2k}-|\psi_{0,2k}^-|m_{2k})$$

$$m_0=z_{0,2k}(|\psi_{0,2k}^-|E_{2k}-\psi_{0,2k}^+m_{2k}).$$

It should be noted that the functions $z_{n-1,n}$ in processes with different number of bounces are different. If $z^{(k)}_{n-1,n}$ refers to the process with $k-1$ bounces, different $z^{(k)}_{n-1,n}$ are connected by the following relations

$$\begin{aligned}
&z^{(k)}_{k-1,k} = z^{(k-1)}_{k-2,k-1} = z^{(k-2)}_{k-3,k-2} = \cdots = z^{(1)}_{01}\,, \\
&z^{(k)}_{k-2,k-1} = z^{(k-1)}_{k-3,k-2} = z^{(k-2)}_{k-4,k-3} = \cdots = z^{(2)}_{01}\,, \\
&\cdots \qquad \cdots \\
&z^{(k)}_{12} = z^{(k-1)}_{01}.
\end{aligned} \tag{63}$$

The distribution function of non-interacting gravitons in the bulk satisfies the Liouville equation without the collision term. If coordinates and momenta gravitons along a geodesic are parametrized by parameter $\lambda$, i.e. $f(x^A(\lambda), p^A(\lambda))$, we have

$$f(R(\lambda_0), p^A(\lambda_0)) = f(R(\lambda_1), p^A(\lambda_1)). \tag{64}$$

In the case of the metric (7) relation (64) can be written as [11]

$$f(\tilde{T}_0, R_0, \tilde{p}^A) = f(\tilde{T}, R_1, \tilde{p}^A R_1/R_0),$$

where $\tilde{p}^A = (\tilde{p}^T, \tilde{p}^R, \mathbf{p})$. If the points $(\tilde{T}_0, R_0)$ and $(\tilde{T}_1, R_1)$ are on the brane world sheet, they are functions of the proper time on the brane. Temperature of the Universe, $T(t)$, is defined through the proper time $t$ via (28)-(29).

### 5.1 First fall of gravitons to the brane

We suppose that the distribution function of emitted gravitons $f^{out}$ depends on $E_0 = \sqrt{m_0^2 + \mathbf{p}^2}$, $m_0$ and temperature $T_0$, i.e. $f^{out} = f^{out}(E_0, m_0, T(t_0))$. The distribution function of gravitons emitted at time $t_0$ and falling back for the first time on the brane at time $t_1$ is

$$f^{in,(1)}(E_1, m_1, T) = f^{out}(E_0(E_1, m_1, z_{01}), m_0(E_1, m_1, z_{01}), T_0(T, z_{01})), \tag{65}$$

where

$$E_0 = z_{01}^{-1}(\psi^+_{01} E_1 + \psi^-_{01} m_1) \tag{66}$$

$$m_0 = z_{01}^{-1}(\psi^-_{01} E_1 + \psi^+_{01} m_1). \tag{67}$$

In the period of early cosmology from (28) and (29) it follows that $T_1/T_0 \simeq \rho_b(t_0)/\rho_b(t_1) = z_{01}$. We obtain the distribution function at time $t_1$ as

$$f^{in,(1)}(E_1, m_1, T) = f^{out}(z_{01}^{-1}(E_1\psi^+(z_{01}) + m_1\psi^-(z_{01})), z_{01}^{-1}(E_1\psi^-(z_{01}) + m_1\psi^+(z_{01})), z_{01}^{-1}T), \tag{68}$$

where $z_{01}$ is determined as a function of $x_1 = m_1/E_1$ by Eq.(45). For $\mu/\rho_1 \ll 1$ the terms $O(\mu/\rho_1)$ could be neglected and $z_{01}$ is defined via (47).

Condition that $x_0 = m_0/E_0 > 0$ takes the form

$$x_0 = \frac{\psi^-(z_{01}) - |x_1|\psi^+(z_{01})}{\psi^+(z_{01}) - |x_1|\psi^-(z_{01})} > 0.$$

Because the nominator of this ratio is positive, this condition is equivalent to $\psi^-(z_{01}) - |x_1|\psi^+(z_{01}) > 0$, or

$$\frac{1+|x_1|}{1-|x_1|} z_{01}^8 < 1.$$

To show that this inequality is satisfied, we wright

$$1 = \frac{1+|x_1|}{1-|x_1|} f_7(z_{01}) = \frac{1+|x_1|}{1-|x_1|} z_{01} \frac{1+\ldots+z_{01}^6}{7} > \frac{1+|x_1|}{1-|x_1|} z_{01}^8. \tag{69}$$

## 5.2 Next falls of gravitons to the brane

The distribution function of gravitons emitted at time $t_0$, which bounced off the brane at time $t_1$ and fall on the brane the second time at time $t_2$ is

$$\begin{aligned} f^{in,(2)}(E_2, m_2, t_2) &= f^{out,(1)}(E_1(E_2, m_2), -m_1(E_2, m_2), t_1) = f^{in,(1)}(E_1(E_2, m_2), m_1(E_2, m_2), t_1) \\ &= f^{out}(E_0(E_1(E_2, m_2), m_1(E_2, m_2)),\ m_0(E_1(E_2, m_2), m_1(E_2, m_2)),\ t_0). \end{aligned} \tag{70}$$

Tracing the propagation of graviton, we obtain

$$\begin{aligned} \begin{pmatrix} E_0 \\ m_0 \end{pmatrix}^{out} &= z_{01}^{-1} \begin{pmatrix} \psi_{01}^+ & \psi_{01}^- \\ \psi_{01}^- & \psi_{01}^+ \end{pmatrix} \begin{pmatrix} E_1 \\ m_1 \end{pmatrix}^{in} \\ = z_{01}^{-1} \begin{pmatrix} \psi_{01}^+ & \psi_{01}^- \\ \psi_{01}^- & \psi_{01}^+ \end{pmatrix} z_{12}^{-1} \begin{pmatrix} \psi_{12}^+ & \psi_{12}^- \\ -\psi_{12}^- & -\psi_{12}^+ \end{pmatrix} \begin{pmatrix} E_2 \\ m_2 \end{pmatrix}^{in} &= (z_{01} z_{12})^{-1} \begin{pmatrix} \psi_{02}^+ & \psi_{02}^- \\ \psi_{02}^- & \psi_{02}^+ \end{pmatrix} \begin{pmatrix} E_2 \\ m_2 \end{pmatrix}^{in}, \end{aligned} \tag{71}$$

where $u_{02} = z_{01} z_{12}^{-1}$ and

$$\psi_{02}^\pm = \frac{1}{2}\left[\psi^+(z_{01})\psi^\pm(z_{12}) - \psi^-(z_{01})\psi^\mp(z_{12})\right] = \frac{1}{2}\left[u_{02}^{-4} \pm u_{02}^4\right].$$

The time $t_1$ of the bounce is determined from Eq. (47) with $z = z_{12} \simeq (t_1/t_2)^{1/4}$,

$$\frac{1-|x_2|}{1+|x_2|} = f_7(z_{12}), \tag{72}$$

where $x_2 = (m_2/E_2)^{in}$. The time $t_0$ is determined from the equation (47) with $z = z_{01} \simeq (t_0/t_1)^{1/4}$

$$\frac{1-|x_1|}{1+|x_1|} = f_7(z_{01}), \tag{73}$$

where $x_1 = (m_1/E_1)^{in}$. Expressing $x_1$ through $x_2$, we have

$$x_1 = x_1^{in} = -\frac{\psi^-(z_{12}) + x_2\psi^+(z_{12})}{\psi^+(z_{12}) + x_2\psi^-(z_{12})} = -\frac{\psi^-(z_{12}) - |x_2|\psi^+(z_{12})}{\psi^+(z_{12}) - |x_2|\psi^-(z_{12})}. \tag{74}$$

Substituting (74) in (73), we obtain

$$f_7(z_{01}) = \frac{1-|x_1|}{1+|x_1|} = \frac{\psi^+(z_{12}) - |x_2|\psi^-(z_{12}) - \psi^-(z_{12}) + |x_2|\psi^+(z_{12})}{\psi^+(z_{12}) - |x_2|\psi^-(z_{12}) + \psi^-(z_{12}) - |x_2|\psi^+(z_{12})} = \frac{1+|x_2|}{1-|x_2|} z_{12}^8. \tag{75}$$

Condition $m_1^{out} > 0$ yields the constraint $\psi^-(z_{12}) - |x_2|\psi^+(z_{12}) > 0$. This condition can be rewritten as $(1 - z_{12}^8)/(1 + z_{12}^8) > |x_2|$, or equivalently, as

$$z_{12}^8 \frac{(1 + |x_2|)}{(1 - |x_2|)} < 1,$$

which is valid, because of (75).

Condition $m_0 > 0$ is $-(\psi_{02}^- - |x_2|\psi_{02}^+) > 0$, or $z_{01}^8(1 - |x_2|) < z_{12}^8(1 + |x_2|)$, which is satisfied, because

$$1 < \frac{1 + |x_2|}{1 - |x_2|} \frac{z_{12}^8}{z_{01}^8} = \frac{f_7(z_{01})}{z_{01}^8}.$$

Using relations (72) and (75) we can show that $z_{12} > z_{01}$.

The distribution function of gravitons (70) cab be expressed as

$$f^{(2)}(E_2, m_2, T) = f^{out,(0)}(z_{02}^{-1}(-E_2\psi_{02}^- - m_2\psi_{02}^+),\ z_{02}^{-1}(E_2\psi_{02}^+ + m_2\psi_{02}^-),\ z_{02}^{-1}T). \tag{76}$$

In the case that gravitons have made two bounces, using (57) and (58), we have

$$\left( \begin{array}{c} E_0 \\ m_0 \end{array} \right)^{out} = z_{03}^{-1} K_0 K_1 L K_1 K_2 L K_2 K_3 \left( \begin{array}{c} E_3 \\ m_3 \end{array} \right)^{in} = z_{03}^{-1} \left( \begin{array}{cc} \psi_{03}^+ & \psi_{03}^- \\ \psi_{03}^- & \psi_{03}^+ \end{array} \right) \left( \begin{array}{c} E_3 \\ m_3 \end{array} \right). \tag{77}$$

Here $z_{03} = z_{01}z_{12}z_{23}$, $\psi_{03}^{\pm} = \psi^{\pm}(u_{03})$ and $u_{03} = z_{01}z_{12}^{-1}z_{23}$. The functions $z_{k-1,k}$ are determined from the equations

$$f_7(z_{23}) = \frac{1 - |x_3|}{1 + |x_3|}, \tag{78}$$

$$f_7(z_{12}) = \frac{1 + |x_3|}{1 - |x_3|} z_{23}^8, \tag{79}$$

$$f_7(z_{01}) = \frac{1 - |x_3|}{1 + |x_3|} \frac{z_{12}^8}{z_{23}^8}. \tag{80}$$

Using the above relations, we can show that $z_{23} > z_{12} > z_{01}$.

The distribution function $f^{(3)}(m_3, E_3, T)$ is

$$f^{(3)}(m_3, E_3, T) = f^{out,(0)}(z_{03}^{-1}(E_3\psi_{03}^- + m_3\psi_{03}^+),\ z_{03}^{-1}(E_3\psi_{03}^+ + m_3\psi_{03}^-),\ z_{03}^{-1}T) \tag{81}$$

# 6 Numerical estimates and discussion

We perform numerical estimates of the $(nn)$ component of the energy-momentum tensor of incoming gravitons using the distribution function of paper [11]. Qualitatively, the energy-momentum tensor of incoming gravitons at the registration time $t_1$ is formed by summing contributions from gravitons emitted at times $t_0$ preceding the registration time $t_1$.

The distribution function of emitted gravitons is

$$f^{(0)}(m, \mathbf{p}, t_0) = Bm^3 e^{-E/T_0}, \qquad B = \frac{A\kappa^2}{2^{10}\pi^5}, \tag{82}$$

where $E = \sqrt{m^2 + \mathbf{p}^2}$, $A$ is the weighted sum of relativistic degrees of freedom which contribute to the annihilation amplitude to gravitons [8]. The $(nn)$ component of the energy-momentum tensor of emitted gravitons, $T^{(em)}_{nn}(t_0)$, is

$$T^{em}_{nn}(t_0) = \int dmd\mathbf{p}\frac{m^2}{2E} f^{(0)}(m, \mathbf{p}, t_0). \tag{83}$$

The $(nn)$ component of the energy-momentum tensor of gravitons falling back to the brane is

$$T^{in,(1)}_{nn}(t_1) = \int dm_1 d\mathbf{p_1}\frac{m_1^2}{2E_1} f^{(1)}(m_1, \mathbf{p_1}, t_1), \tag{84}$$

where the distribution function of infalling gravitons is

$$f^{in,(1)}(m_1, \mathbf{p_1}, t_1, T) = Bm_0^3(m_1, E_1, z_{01}) \exp\left\{-\frac{E_0(m_1, E_1, z_{01})}{T_0(T, z_{01})}\right\}. \tag{85}$$

$E_0(m_1, E_1, z_{01})$ and $m_0(m_1, E_1, z_{01})$ are defined by (66)-(67). Substituting these expressions, we have

$$\begin{aligned} T^{in,(1)}_{nn} &= 2\pi B \int_{-E_1}^{0} dm_1\, m_1^2 \sqrt{E_1^2 - m_1^2} \int dE_1 z_{01}^{-3}(E_1\psi^-_{01} + m_1\psi^+_{01})^3 \exp\left\{-\frac{E_1\psi^+_{01} + m_1\psi^-_{01}}{T}\right\} \\ &= 2\pi B \int_{0}^{E_1} dm_1\, m_1^2 \sqrt{E_1^2 - m_1^2} \int dE_1 z_{01}^{-3}(E_1\psi^-_{01} - m_1\psi^+_{01})^3 \exp\left\{-\frac{E_1\psi^+_{01} - m_1\psi^-_{01}}{T}\right\} \end{aligned} \tag{86}$$

Introducing $x = m_1/E_1$, we express $T^{(1)}_{nn}$ as

$$T^{in,(1)}_{nn} = 2\pi B \int dx x^2\sqrt{1 - x^2} \int dE_1 E_1^7(\psi^-_{01} - x\psi^+_{01})^3 z_{01}^{-3}(x) \exp\left\{\frac{-E_1(\psi^+_{01} - x\psi^-_{01})}{T}\right\}. \tag{87}$$

First, we integrate over $E_1$ in the limits $(0, \infty)$, and below we consider integration taking into account lower and upper bounds. For $T^{(1)}_{nn}$ we have

$$T^{in,(1)}_{nn}(T) = 2\pi BT^8\Gamma(8) \int_0^1 dx x^2\sqrt{1 - x^2} z_{01}^{-3}(x) \frac{(\psi^-_{01} - x\psi^+_{01})^3}{(\psi^+_{01} - x\psi^-_{01})^8} \tag{88}$$

For the energy-momentum tensor of gravitons which have made one bounce we obtain

$$T^{in,(2)}_{nn} = 2\pi^2 B \int_0^\infty dE_2 \int_{-E_2}^{0} dm_2\, m_2^2 \sqrt{E_2^2 - m_2^2}\, z_{02}^{-3}(-E_2\psi^-_{02} - m_2\psi^+_{02})^3 \exp\left\{-\frac{E_2\psi^+_{02} + m_2\psi^-_{02}}{T}\right\}, \tag{89}$$

where $z_{02} = z^{(2)}_{12}\, z^{(2)}_{01}$. From the inequality $z_{12} > z_{01}$ it follows that $\psi_{02}(z^{(2)}_{12}/z^{(2)}_{01}) < 0$. Instead, we use $|\psi_{02}| = \psi_{02}(z^{(2)}_{01}/z^{(2)}_{12})$. $T^{in,(2)}_{nn}$ is expressed as

$$T^{in,(2)}_{nn} = 2\pi BT^8\Gamma(8) \int_0^1 dx\, x^2\sqrt{1 - x^2}\, z_{02}^{-3} \frac{(|\psi^-_{02}| + x\psi^+_{02})^3}{(\psi^+_{02} + x|\psi^-_{02}|)^8} \tag{90}$$

For the energy-momentum tensor of gravitons which have made two bounces we have

$$\begin{aligned} T^{in,(3)}_{nn} &= 2\pi B \int_0^\infty dE_3 \int_{-E_3}^{0} dm_2\, m_3^2 \sqrt{E_3^2 - m_3^2}\, z_{03}^{-3}(E_3\psi^-_{03} + m_3\psi^+_{03})^3 \exp\left\{-\frac{E_3\psi^+_{03} + m_3\psi^-_{03}}{T}\right\} \\ &= 2\pi BT^8\Gamma(8) \int_0^1 dx\, x^2\sqrt{1 - x^2}\, z_{03}^{-3} \frac{(\psi^-_{03} - x\psi^+_{03})^3}{(\psi^+_{03} - x\psi^-_{03})^8}, \end{aligned} \tag{91}$$

where $z_{03} = z_{23}^{(3)} z_{12}^{(3)} z_{01}^{(3)}$.

Integration over $E$ in the integrals $T_{nn}^{in,(k)}$ is performed for $E > T_{min}$.

Assuming that $T_{min}$ is in the region of early cosmology, i.e. $10 \lesssim \rho(T_{min})/\mu$, taking $g_* \sim 200$ [2] and using (27), we find that

$$T_{min}^4 \sim \frac{\mu}{\kappa^2}. \tag{92}$$

Using the relation $M^3 \simeq \mu M_{pl}^2$ which follows from the fit of cosmological data [4], we have $T_{min}^4 \sim (\mu M_{pl})^2/8\pi$. For $\mu = 10^{-13} \div 10^{-9} GeV$ condition (92) yields $T_{min} \sim 10^3 \div 10^5 GeV$.

Because of the high power of $E$ in the integrals for $T^{(k),in}$, the main contribution to the integrals is produced from the region near the upper limit of integration $T_{max}$. Provided $T_{min} \ll T_{max}$, we set $T_{min} = 0$. The functions $z_{k-1,k}(x)$ and the integrands $I^{(k)}$ for characteristic values of $x$ are given in Table 1 and Fig. 1. For the following it is convenient to introduce the notations

$$T_{nn}^{in,(k)} = 2\pi B T^8 \Gamma(8) \int dx I^{(k)}(x). \tag{93}$$

Table 1: The functions $z_{k-1,k}^{(3)}$ for different values of $x$.

| $x_3 \equiv x$ | .2 | .3 | .4 | .5 | .6 | .7 | .8 |
|---|---|---|---|---|---|---|---|
| $z_{01}^{(3)}(x)$ | .870 | .775 | .633 | .399 | .112 | .0034 | $< 10^{-4}$ |
| $z_{12}^{(3)}(x)$ | .887 | .816 | .729 | .617 | .468 | .267 | .079 |
| $z_{23}^{(3)}(x)$ | .899 | .845 | .787 | .722 | .648 | .557 | .438 |

[2]The ambiguity in $g_*$ and in $A$ in (82) is due to incomplete knowledge of the contribution of dark matter. We assume that the mass of particles which form dark matter is in the interval $(20 \div 100) GeV$. In the period of early cosmology these particles are relativistic. Phenomenologically the acceptable number of dark matter particles with the mass in the above interval is $g_* \sim 100$ [19]. With the number of particle species in the non-supersymmetric Standard model $g_* \sim 100$, the total number is $\sim 200$. Because of the high power of $T$ estimates weakly depend on variations of this number.

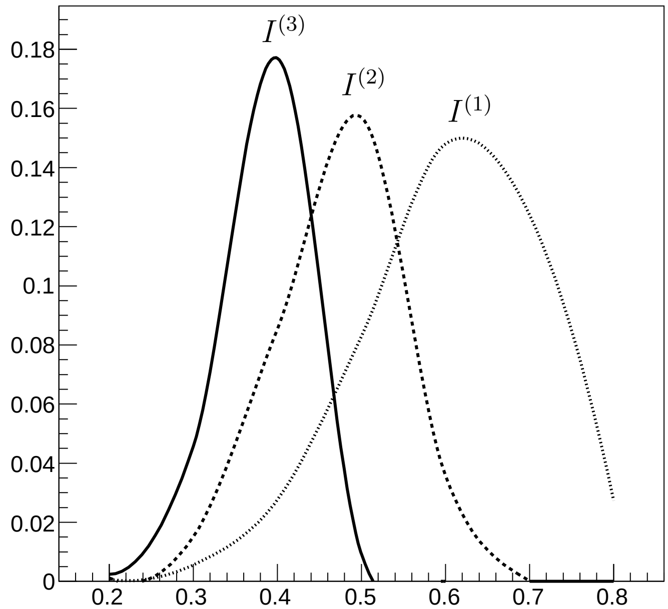


Figure 1: The integrands $I^{(k)}(x),\ k=1,2,3$ as the functions of $x \equiv x_3$ calculated for $T_R/T \gg 1$.

The integrals $T_{nn}^{in,(k)}$ were calculated numerically by substituting the values for $z_{ij}(x)$. For the integrals $\int dx I^{(k)}$ we obtain

$$\int_0^1 dx I^{(1)} = 0.0415, \qquad \int_0^1 dx I^{(2)} = 0.0295, \qquad \int_0^1 dx I^{(3)} = 0.0023. \tag{94}$$

To study dependence of $z(x)$ on $\mu/\rho_1$, we calculate $z(x)$ making use of (45) and taking into account the next order in $\mu/\rho_1$. In Fig.2 the results for $I^{(1)}(x)$ are compared for two values of $\mu/\rho_1 = 0.1$ and $0.01$. Although the values of $z(x)$ for $\mu/\rho_1 = 0.1$ and $\mu/\rho_1 = 0.01$ considerably differ, the values of integrals $\int dx I^{(1)}(\mu/\rho_1 = 0.1) = 0.0460$ and $\int dx I^{(1)}(\mu/\rho_1 = 0.01) = 0.0415$ are close.

The limiting temperature at which the emission begins is the reheating temperature $T_R$. The emission energy $E_0$ is bounded by $T_R$. Using the expression of Sect.5 $E_0 = z_{0n} E(\psi_{0n}^+ - x\psi_{0n}^-)$, we have

$$E_{max} = T_R z_{0n}/(\psi_{0n}^+ - x\psi_{0n}^-),$$

where $z_{0n}(x)$ and $\psi_{0n}^{\pm} = \psi^{\pm}(u_{0n})$ are defined in (59) and (60) correspondingly. For $T_{nn}^{n,(k)}$ we obtain

$$T_{nn}^{in,(n)} = 2\pi B T^8 \int_0^1 dx x^2 \sqrt{1-x^2} z_{0n}^{-3}(x) \frac{(\psi^-(z_{0n}) - x\psi^+(z_{0n}))^3}{(\psi^+(z_{0n}) - x\psi^-(z_{0n}))^8} \gamma(8, z_{0n}(x) T_R/T), \tag{95}$$

where $\gamma$ is incomplete gamma-function.

If $T/T_R \lesssim 1$, the region producing the main contribution to the integral is $1 > z_{0n} > T/T_R$. The corresponding region of $x$ is $0 < x < O(1 - T/T_R)$, where $1 - T/T_R \ll 1$. At small $x$ the integrand decreases as a power of $x$, and contribution from this region is strongly suppressed.

If $T/T_R \ll 1$, the region producing the main contribution to the integral is $1 > z_{0n} > 1 - T/T_R$, where $1 - T/T_R \lesssim 1$.

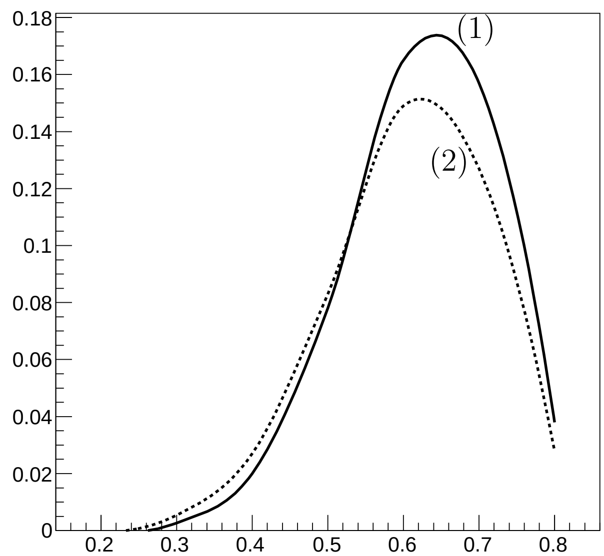


Figure 2: Dependence of $I^{(1)}(x)$ on $\mu/\rho_1, \quad \rho_1 = \rho(t_1)$, where $t_1$ is the time of the first return of graviton to the brane calculated using (46). The function $I^{(1)}(x)$ calculated for $\mu/\rho_1 = 0.01$ (curve (1)) and $\mu/\rho_1 = 0.1$ (curve (2)). For $\mu/\rho_1 < 0.01$ the values of $I^{(1)}(x)$ are very close to the case with $\mu/\rho_1 = 0$.

The energy density of dark radiation satisfies the evolution equation [9, 11, 13]

$$\frac{d\rho_D}{dt} + 4H\rho_D \simeq -\frac{2\rho}{\mu}(T^{em}_{vn} + T^{em}_{nn} - T^{in}_{nn}), \tag{96}$$

where $T^{in} = \sum T^{in,(k)}$ and

$$T^{em}_{vn}(T) = -2\pi B\, T^8 \Gamma(8) \cdot \frac{\pi}{32} \frac{\gamma(8, T_R/T)}{\Gamma(8)}, \tag{97}$$

$$T^{em}_{nn}(T) = 2\pi B\, T^8 \Gamma(8) \cdot \frac{8}{105} \frac{\gamma(8, T_R/T))}{\Gamma(8)}. \tag{98}$$

In the period of early cosmology from the Friedmann and approximate conservation equations, $\dot{\rho} + 4H\rho \simeq 0$, it follows that

$$\frac{\rho}{\mu} = \frac{1}{4\mu t} = \frac{2\kappa^2 \pi^3 g_*(T) T^4}{45(\mu\, M_{pl})^2}.$$

Here we substituted $M_5^3 \simeq \mu M_{pl}^2$ [4]. Eq. (96) is transformed as

$$\frac{d\rho_D(T)}{dT} - \frac{4}{T}\rho_D(T) = \frac{2}{\mu T}(T^{em}_{vn} + T^{em}_{nn} - T^{in}_{nn}). \tag{99}$$

Explicitly we have

$$\frac{d\rho_D}{dT} - \frac{4}{T}\rho_D = 4\pi T^7 \Gamma(8) B \left[ \left( -\frac{\pi}{32} + \frac{8}{105} \right) \frac{\gamma(8, T_R/T)}{\Gamma(8)} - \sum \int dx I^{(k)}(x) \frac{\gamma(8, Z_{0k} T_R/T}{\Gamma(8)} \right] \tag{100}$$

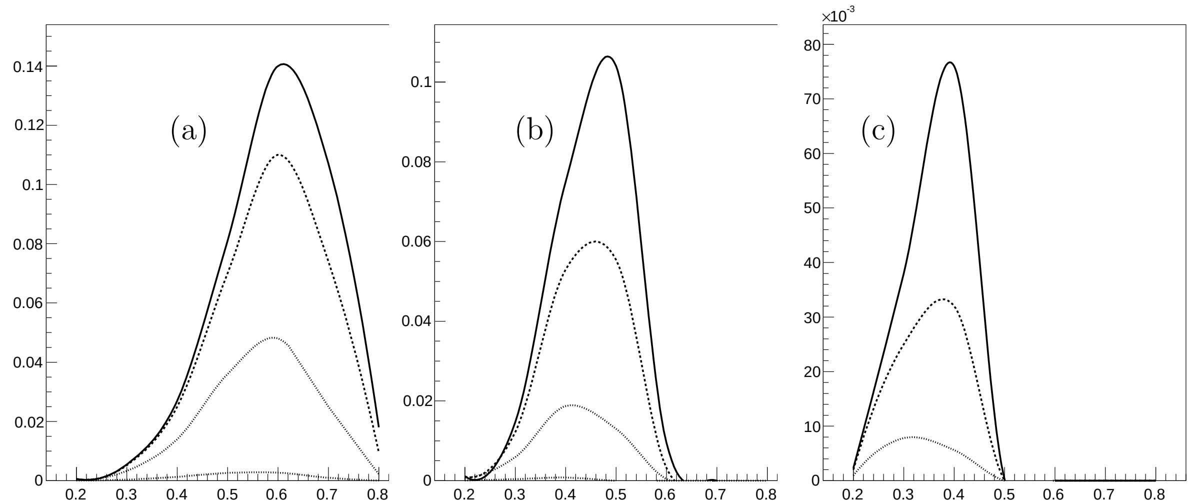


Figure 3: The functions $I^{(1)}(x)\gamma(8, z_{01}(x)T_R/T)$ calculated for $T_R/T = 20, 15, 10, 5$ (curves from top to bottom) (plot (a)). The same for $I^{(2)}(x)\gamma(8,(z_{01}z_{12})(x)T_R/T)$ (plot (b)) and $I^{(3)}(x)\gamma(8,(z_{01}z_{12}z_{23})(x)T_R/T)$ (plot (c)). It is seen that as $k$ increases the region of $x$, which produces the main contribution to $T^{in,(k)}$ is shifted to smaller $x$ and larger $z(x)$.

Integrating (100) with the boundary condition $\rho_D(T_R) = 0$, we obtain

$$\rho_D = -T^4 \int\limits_T^{T_R} dT'\, T'^3 \frac{\Gamma(8)A\kappa^2}{\mu 2^8 \pi^4} \left[ \left( -\frac{\pi}{32} + \frac{8}{105} \right) \frac{\gamma(8, T_R/T')}{\Gamma(8)} - \sum \int dx I^{(k)}(x) \frac{\gamma(8, Z_{0k}T_R/T')}{\Gamma(8)} \right] \tag{101}$$

For the ratio $\rho_D/\hat{\rho}$ we have

$$\begin{aligned} \frac{\rho_D(T)}{\hat{\rho}(T)} &= \frac{4725\, A}{2^5\pi^4 g_*(T)(\mu M_{pl})^2} \int_T^{T_R} \left[ -0.224 \frac{\gamma(8, T_R/T')}{\Gamma(8)} + \frac{32}{\pi} \sum \int dx I^{(k)}(x) \frac{\gamma(8, Z_{0k}T_R/T')}{\Gamma(8)} \right] T'^3 dT' \\ &= \frac{4725\, A\, T_R^4}{2^5\pi^4 g_*(T)(\mu M_{pl})^2} \int_1^{T_R/T} \frac{dy}{y^5} \left[ -0.224 \frac{\gamma(8, y)}{\Gamma(8)} + \frac{32}{\pi} \sum \int_0^1 dx I^{(k)}(x) \frac{\gamma(8, Z_{0k}y)}{\Gamma(8)} \right] \end{aligned} \tag{102}$$

Here we substituted $M_5^3 \simeq \mu M_{pl}^2$ [4].

For $T_R/T > 20$ the integral is practically constant and independent of $T$. The main contribution to the integral is produced by integration over the region of $y$ near the lower limit. Taking $T_R/T = 20$ and performing integration over $y$, we obtain

$$0.224 \int_1^{T_R/T} \frac{dy}{y^5} \frac{\gamma(8, y)}{\Gamma(8)} \simeq 6.56 \cdot 10^{-5} \tag{103}$$

$$\frac{32}{\pi} \int_1^{T_R/T} \frac{dy}{y^5} \sum_1^3 \int dx I^{(k)}(x) \frac{\gamma(8, z_{0k}y)}{\Gamma(8)} \simeq 3.8 \cdot 10^{-5}. \tag{104}$$

For the ratio of energy density of dark radiation to energy density of matter we have

$$\left| \frac{\rho_D}{\hat{\rho}} \right| \simeq 3.5 \cdot 10^{-5} \frac{T_R^4}{(\mu M_{pl})^2}. \tag{105}$$

A typical order of constraint on magnitude of the ratio $\rho_D/\hat{\rho}$ in the period of early cosmology, which follows from primordial nucleosynthesis, is $|\rho_D/\hat{\rho}| \lesssim 0.07$ [20].

From the gravity experiments it follows that characteristic scale of extra dimension $r_{extr} \sim \mu^{-1}$ is less than $10^{-2}cm$, or $\mu > 2 \cdot 10^{-12} GeV$ [1]. For $\mu \sim 10^{-12}\, GeV$ the estimate (105) gives $T_R \sim 2.3 \cdot 10^4\, GeV$. This value of $T_R$ is significantly lower than usually accepted $T_R \sim 10^5 \div 10^7\, GeV$, indicating that large extra dimensions can appear with low reheating temperature. The estimate can be improved, if there is more complete cancellation between two terms in (102), or for larger values of $\mu$. Because of strong dependence of $\rho_D/\hat{\rho}$ on $\mu$, the possibility of larger $\mu$ seems more plausible. For the reheating temperature $T_R \sim 10^6 GeV$ the scale of extra dimension obtained from (105) is $\mu \sim 2 \cdot 10^{-9} GeV$ ($\mu^{-1} \sim 10^{-5}cm$). For higher reheating temperatutes the value of $\mu$ rapidly increases: $\mu \sim T_R^2$. For larger $\mu$, in the integrals for $T_{nn}^{(k)}$, the lower bound of integration $E_{min}$ increases (see (92)) resulting in smaller magnitudes of the integrals $T_{nn}^{in}$. For $T_R > 10^6 GeV$ this does not change the above results significantly, but for smaller $T_R$ the effect of the lower bound of integration must be taken into account.

The result of our calculations showing that account of gravity radiation leads to high mass scale of extra dimension can be attributed either to an insufficient accuracy of calculations (although our tests indicate stability of the result), or indicate that "too large" extra dimensions are incompatible with this class of models.

# 7 Conclusion

In this paper in a model with extra dimension we have calculated graviton emission to the extra dimension. Graviton emission is significant at the high-temperature period of the evolution of the Universe. The key point of the present paper is solution of the system of equations for the brane trajectory and for geodesic equation for graviton trajectory. For a given value of graviton 5-momentun at the time of graviton detection, we calculated the time of graviton emission. We obtained the recursion relations enabling, in principle, calculate the energy-momentum tensor of gravitons falling back to the brane which have made an arbitrary number of bounces. For the first three returns of graviton to the brane we obtained the explicit expressions for the energy-momentum tensor of the gravitons falling back to the brane and made their numerical estimates. Solving the evolution equation for the energy density of the dark radiation, we obtained a relation connecting the reheating temperature and the scale of extra dimension and estimated the scale of the extra dimension.


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